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Cell-Centered Multigrid for Interface Problems

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A multigrid method is presented for cell-centered discretizations of elliptic partial differential equations. The method works both for smooth and strongly discontinuous coefficients, even though, in contrast with earlier works, the prolongation and restriction operators do not depend on the equation. © 1988 Academic Press, Inc.

1. INTRODUCTION

The multigrid method to be presented will be developed for the equation

$$-\frac{\partial}{\partial x}\left(a\frac{\partial \phi}{\partial x}\right)-\frac{\partial}{\partial y}\left(a\frac{\partial \phi}{\partial y}\right)=f,$$
$$(x, y)\in\Omega=(0,1)\times(0,1), \qquad \phi|_{\partial\Omega}=g, \qquad a>0. \tag{1.1}$$

The coefficient $a(x, y)$ is not continuous everywhere. This precludes application of standard multigrid methods. Alcouffe *et al.* [1], Dendy [3], Kettler and Meijerink [6] (see also Kettler [7]) have developed special multigrid methods that work well for the problem considered here. In these methods the prolongation and restricton operators depend on the discrete approximation to (1.1). Until now, theoretical justification is lacking and seems hard to come by. In the following, a multigrid method is proposed for (1.1) that also works in practice, is simpler, and can be justified theoretically. The difference with the methods just mentioned is that prolongation and restriction are not problem-dependent and that grid coarsening is done cell-wise rather than point-wise. What this means will be made clear in the sequel.

2. FINITE VOLUME DISCRETIZATION

For convenience, the mesh size will be h in both directions. The domain Ω is subdivided in finite volumes or cells, which are squares of size h, with centers at the points

$$\Omega_h=\{(x, y): x=x_i=(i-\tfrac{1}{2})h,\ y=y_j=(j-\tfrac{1}{2})h;\ i, j=1, 2, ..., n;\ h=1/n\}. \tag{2.1}$$

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The cell with center at (x_i, y_j) is denoted by Ω_{ij}, and ϕ_{ij} is the value of ϕ at the center. Often, this is called a block-centered or cell-centered grid. Forward and backward divided differences in x- and y-direction are defined by

$$\Delta_x \phi_{ij} = (\phi_{i+1,j} - \phi_{ij})/h, \qquad \nabla_x \phi_{ij} = (\phi_{ij} - \phi_{i-1,j})/h, \tag{2.2}$$

and similarly for Δ_y and ∇_y.

For completeness we briefly review the elementary aspects of finite volume discretization of (1.1) with discontinuous coefficient a. Equation (1.1) is integrated over the finite volume Ω_{ij}. With the Gauss divergence theorem this results in

$$-\int_{\partial\Omega_{ij}} a\phi_{,\alpha} n_\alpha \, d\Gamma = \int_{\Omega_{ij}} f \, d\Omega = h^2 f_{ij}, \tag{2.3}$$

with the summation convention for the index α.

Let S_{ij}^{α} be the side of Ω_{ij} with outward normal in the x_α-direction ($x_1 = x$, $x_2 = y$). Equation (2.3) can be rewritten as

$$-\nabla_\alpha F_\alpha^{ij} = h f_{ij}, \tag{2.4}$$

with the flux F_α^{ij} defined as

$$F_\alpha^{ij} = \int_{S_{ij}^{\alpha}} a\phi_{,\alpha} \, d\Gamma. \tag{2.5}$$

We discuss the approximation of F_1^{ij}; F_2^{ij} is treated similarly. F_1^{ij} is approximated as

$$F_1^{ij} \simeq h a_{ij} \frac{\partial \phi}{\partial x}(x_i + h/2, y_i), \tag{2.6}$$

where a_{ij} is the average of a over Ω_{ij}. The approximation

$$\frac{\partial \phi}{\partial x}(x_{ij} + h/2, y_{ij}) \simeq \Delta_x \phi_{ij} \tag{2.7}$$

is out of the question, since a_{ij} may differ strongly between adjacent cells, so that $\partial\phi/\partial x$ may have large jumps at cell boundaries. A correct approximation is obtained as follows. Point of departure is that ϕ and $a\,\partial\phi/\partial x$ are continuous. Denote for brevity $\phi(x_{ij} + h/2, y_{ij})$ by ϕ^*. Then we approximate F_1^{ij} by

$$F_1^{ij} \simeq 2a_{ij}(\phi^* - \phi_{ij}) = 2a_{i+1,j}(\phi_{i+1,j} - \phi^*). \tag{2.8}$$

Elimination of ϕ^* from (2.8) results in

$$F_1^{ij} \simeq h w_{ij}^x \Delta_x \phi_{ij} \tag{2.9}$$

with

$$w_{ij}^x = 2a_{ij}a_{i+1,j}/(a_{ij} + a_{i+1,j}). \tag{2.10}$$

Similarly, we obtain

$$F_2^{ij} \simeq h w_{ij}^y \Delta_y \phi_{ij} \tag{2.11}$$

with

$$w_{ij}^y = 2a_{ij}a_{i,j+1}/(a_{ij} + a_{i,j+1}). \tag{2.12}$$

Substitution of (2.9) and (2.11) in (2.4) results in

$$-(\nabla_x w^x \Delta_x + \nabla_y w^y \Delta_y)\phi = f. \tag{2.13}$$

It is easy to see that w^x and w^y satisfy

$$\inf(a) \leqslant w^x, w^y \leqslant \sup(a). \tag{2.14}$$

The Dirichlet boundary condition is implemented as follows. Consider the side $x = 0$. There F_1^{0j} is approximated by (cf. (2.8)):

$$F_1^{0j} \simeq 2a_{1j}(\phi_{1j} - g_j). \tag{2.15}$$

A Neumann boundary condition gives F_1^{0j} directly.

3. Prolongation and Restriction

The reader is assumed to be familiar with multigrid methods. For an introduction, see, for example, Hackbusch and Trottenberg [4], Hackbusch [5], or McCormick [8].

Coarse grids are constructed cell-wise. That is, coarser grids $\Omega_{2h}, \Omega_{4h}, \ldots$ are obtained by successively doubling h in (2.1). Hence, each coarse cell is the union of four finer cells. The cell centers of a coarse grid do not belong to the next finer grid. This is different from point-wise coarsening, where coarse grids are constructed by deleting grid points, so that coarse grid points always belong to a finer grid.

The grid with mesh size h is denoted by Ω_h, and $\Phi_h: \Omega_h \to \mathbb{R}$ is the corresponding set of grid functions. Elements of Φ_h are denoted by ϕ^h, ψ^h.

In this section the choice of prolongation and restriction operators

$$P_h: \Phi_{2h} \to \Phi_h, \qquad R_{2h}: \Phi_h \to \Phi_{2h}$$

is discussed. One possibility is

$$\begin{aligned}(P_h\phi^{2h})_{2i,2j} &= (P_h\phi^{2h})_{2i-1,2j} = (P_h\phi^{2h})_{2i,2j-1} \\ &= (P_h\phi^{2h})_{2i-1,2j-1} = \phi_{ij}^{2h}.\end{aligned} \tag{3.1}$$

A possibility for R_{2h} is

$$R_{2h} = P_h^*, \tag{3.2}$$

with superscript $*$ denoting the adjoint. With the inner product

$$(\phi^h, \psi^h) = h^2 \sum_{\Omega_h} \phi_{ij}^h \psi_{ij}^h \tag{3.3}$$

we find that the stencil of R_{2h} defined by (3.1), (3.2) is

$$[R_{2h}] = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \tag{3.4}$$

where $[\cdot]$ denotes the stencil of the corresponding operator.

P_h and R_{2h}^* interpolate polynomials exactly of degree at most 0. Their order m_P, m_R is defined to be the maximum degree of exactly interpolated polynomials plus 1; hence for (3.1), (3.2) we have

$$m_P = m_R = 1. \tag{3.5}$$

We must have

$$m_P + m_R > 2m \tag{3.6}$$

(Brandt [2], Hackbusch [5]), with $2m$ the order of the differential equation to be solved. Hence, (3.1), (3.2) are not right for (1.1). See Wesseling [10] for what happens when one does use (3.1), (3.2) for (1.1).

A restriction with $m_R = 2$ is given by

$$[R_{2h}] = \frac{1}{16}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 3 & 2 & 0 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}. \tag{3.7}$$

At the boundaries, (3.7) has to be modified. For a Dirichlet boundary condition we obtain at the boundary $y = 1$ or at the boundary $x = 0$,

$$[R_{2h}] = \frac{1}{16}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \tag{3.8}$$

and similarly at other parts of the boundary. This restriction is obtained as adjoint of linear interpolation. For simplicity, (3.8) is also used in the case of Neumann boundary conditions.

Let the system of equations to be solved on Ω_h be denoted as

$$A_h \phi^h = f^h. \tag{3.9}$$

On Ω_{2h}, A_h is approximated by

$$A_{2h} = R_{2h} A_h P_h. \tag{3.10}$$

Let A_h have a 7-point stencil:

$$[A_h] = \begin{bmatrix} * & * & 0 \\ * & * & * \\ 0 & * & * \end{bmatrix}. \tag{3.11}$$

Then it is found that with P_h, R_{2h} given by (3.1) and (3.7)–(3.8), A_{2h} as given by (3.10) also has a 7-point stencil. This is also true if P_h^* is given by (3.7)–(3.8) and R_{2h} by (3.4). However, if both P_h^* and R_{2h} are given by (3.7)–(3.8) then the stencil of A_{2h} is larger than that of A_h. Therefore it was decided to choose R_{2h} according to (3.7)–(3.8) and P_h according to (3.1). Note that we have $m_P + m_R = 3$, which suffices.

It is easy to obtain A_{2h} explicitly from (3.10), with the choice just made for P_h and R_{2h}. It is found that A_{2h} corresponds to the following discrete equation on the coarse grid (cf. Eq. (2.13)),

$$-(\bar{\nabla}_x \bar{w}^x \bar{A}_x + \bar{\nabla}_y \bar{w}^y \bar{A}_y)\bar{\phi} = \bar{f}, \tag{3.12}$$

where quantities belonging to the coarse grid are denoted by an overbar. We find the simple relation for $\bar{w}^x$, $\bar{w}^y$,

$$\bar{w}^x_{ij} = \tfrac{1}{2}(w^x_{2i,2j} + w^x_{2i,2j-1}), \qquad \bar{w}^y_{ij} = \tfrac{1}{2}(w^y_{2i,2j} + w^y_{2i-1,2j}). \tag{3.13}$$

Hence, in this case construction of coarse grid matrices by (3.10) (Galerkin approximation) is extremely cheap. Note that A_{2h} is symmetric.

4. Numerical Experiments

The multigrid schedule used is the W-cycle with one post-smoothing iteration. The smoothing method is the *ILU*-method described in Wesseling (1982, 1987).

The test problems are the interface problems sketched in Fig. 4.1. In the first problem we have two concentric squares, in the second problem the inner square is rotated over 45°. The sides of the outer square have length 1, of the inner cell nh in problem 1, and $nh/\sqrt{2}$ in problem 2. In the inner square we have $a = a_1 = 0.333 * 10^5$, in the outer square $a = a_2 = 2$. Problem 3 was suggested by Achi Brandt. The cells with centers at $x = (n - 1/2)h$ constitute a vertical isolating strip of width h, where the value of the diffusion coefficients is $a = a_1 = 10^{-10}$; outside the strip, $a = a_2 = 2$.

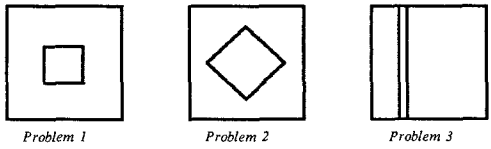

FIG. 4.1. Geometry of test problems.

We solve (1.1) with $f=xy$, $g=x^2+y^2$, starting iterand zero. Twelve iterations were carried out. The average reduction factor ρ is defined as

$$\rho=\{\|r^m\|/\|r^o\|\}^{1/m} \tag{4.1}$$

with $\|\cdot\|$ the l_2-norm, r the residue $r=b^h-A_h\phi^h$ on the finest grid (with $A_h\phi^h=b^h$ the system top be solved), r^o the initial residue, r^m the final residue, and m the number of multigrid iterations carried out. Table I gives ρ for a number of cases. Where $n\neq 0$ we have taken the worst case for all $0\leqslant n\leqslant h^{-1}$. The last column is for Neumann boundary conditions along $x=0$ and $y=0$.

It is clear that multigrid works efficiently. For problems 1 and 2, ρ does not increase with h. With $n\neq 0$, ρ is larger than with $n=0$ (Poisson equation). We think this is due to the fact that the equations in the inner square are almost uncoupled from those outside for $a_1\gg a_2$, so that we almost have a discretized pure Neumann problem for the interior square, which is singular. This hypothesis is confirmed by the fact that with a_1 and a_2 interchanged ($a_1\ll a_2$), ρ is found to be about the same size for all n, including 0. For problem 3, ρ increases with h for certain locations of the isolating strip. This is thought to be due to the fact that, as suggested by Achi Brandt, according to Eq. (3.13) the isolation (small value of w) between the regions separated by the vertical strip may disappear after two coarsenings. Nevertheless, convergence is still rapid. Inspection of the last column of Table I shows that the introduction of Neumann boundary conditions has little influence. Therefore it does not seem worthwhile to abandon (3.8) along non-Dirichlet boundaries.

With another smoothing method, namely point Gauss–Seidel, similar results were obtained.

TABLE I

n, ρ for Problems 1, 2, and 3

Problem\h^{-1}	8		16		32		64		64	
1	0	0.059	0	0.077	0	0.085	0	0.091	0	0.090
1	6	0.312	10	0.362	26	0.304	58	0.290	58	0.298
2	6	0.245	10	0.300	26	0.273	58	0.237	58	0.220
3	1	0.061	10	0.074	18	0.147	34	0.299	34	0.372

5. Discussion

Multigrid methods that work for elliptic equations with discontinuous coefficients (interface problems) have been described by Alcouffe *et al.* [1], Dendy [3], Kettler and Meijerink [6], Kettler [7], and in the present work. The present method differs from the earlier ones in that grid coarsening is done cell-wise rather than point-wise, and prolongation and restriction are not dependent on the equations. As a result, the present method is simpler and requires less storage.

Comparing the rates of convergence that are reported one gets the impression that the present method is at least as efficient as the earlier ones.

Thanks to the simplicity of the present method, it can be jusified theoretically. The theory will be given elsewhere. Why does the present method work? An important factor probably is that (3.12) is quite similar to (2.13). This suggests that the present prolongation and restriction result in accurate coarse grid approximation. Also, the similarity between (3.12) and (2.13) simplifies the theory.

Extension to 3D seems easier than for the older methods. The same considerations as for the older methods are expected to apply to the extension to systems of differential equations.

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